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On analytic and coanalytic function spaces $C_p(X)$

Witold Marciszewski*

Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland

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Abstract

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Under the assumption $(V=L)$ we construct countable completely regular spaces X and Y such that the spaces $C_p(X)$ and $C_p(Y)$ of real-valued continuous functions on X and Y, equipped with the pointwise convergence topology, are analytic noncoanalytic and they are not homeomorphic. We also give analogous examples of coanalytic nonanalytic function spaces.

Keywords: Function space, pointwise convergence topology, analytic set, coanalytic set.

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1. Introduction

For a completely regular space X by $C_p(X)$ we denote the space of all continuous real-valued functions on X endowed with the topology of pointwise convergence. If X is infinite and countable then the space $C_p(X)$ is a dense linear subspace of the countable Cartesian product of the real line $\mathbb{R}^X$. This paper is related to the general problem of classification of Borel and projective spaces $C_p(X)$ for countable X (we refer the reader to [4] for information about Borel and projective classes, here we only recall that the analytic sets, i.e., the sets of the projective class 1, are the continuous images of the space of irrationals and the coanalytic sets, i.e., the sets of the projective class 2, are the complements of analytic sets in separable completely metrizable spaces). In [2] it has been conjectured that for infinite countable X the Borel class of $C_p(X)$ determines its topological type and this

Correspondence to: Dr. W. Marciszewski, Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland. E-mail: wmarcisz@mimuw.edu.pl

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conjecture has been proved for Borel classes not higher than 2. We are not aware of any examples of nonhomeomorphic function spaces $C_p(X)$ and $C_p(Y)$ of the same Borel class. In this paper we prove the following:

Theorem 1.1. *Assume that there exists an uncountable coanalytic subset of the Cantor set 2^ω which does not contain any perfect subset, i.e., a copy of 2^ω. Then there exist countable completely regular spaces X_1, Y_1, X_2 and Y_2 such that*

(a) *the function spaces $C_p(X_1)$ and $C_p(Y_1)$ are analytic noncoanalytic and they are not homeomorphic,*

(b) *the function spaces $C_p(X_2)$ and $C_p(Y_2)$ are coanalytic nonanalytic and they are not homeomorphic.*

Let us note that such coanalytic subsets of 2^ω exist in Gödel's constructible universe L, see [3, Corollary 2, p. 529] or [6, Chapter 5].

In the next section we will describe a general process by which, in Section 5, the spaces X_i and Y_i will be constructed. Sections 3 and 4 contain some auxiliary results.

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2. Function spaces associated with filters

The spaces X_i and Y_i to be constructed in Section 5 are spaces with exactly one nonisolated point, associated in a standard way with filters on a countable set T. Here, we discuss only filters on T containing all cofinite subsets of T and we consider filters to be subspaces of the Cantor set 2^T.

Let F be a filter on the countable set T. The space N_F is the set $T \cup \{\infty\}$ topologized by isolating the points of T and by using the family $\{A \cup \{\infty\}: A \in F\}$ as a neighborhood base at ∞.

For technical reasons it will be more convenient to consider the following space C_F instead of $C_p(N_F)$:

$$C_F = \{f \in \mathbb{R}^T: (\forall \varepsilon > 0)(\exists A \in F)[|f(t)| \leq \varepsilon \text{ for all } t \in A]\}.$$

We have the following simple fact:

Lemma 2.1. *For every filter F on the countable set T the spaces $C_p(N_F)$ and C_F are homeomorphic.*

Proof. One can easily verify that the space $C_p(N_F)$ is linearly homeomorphic to the product $\mathbb{R} \times C_F$. We shall consider two cases:

Case 1: *F contains only cofinite subsets of T.* It is standard that in this case $\mathbb{R} \times C_F$ is homeomorphic to C_F.

Case 2: *F contains a set A such that $B = T\setminus A$ is infinite.* Then for the filter $G = \{C \subseteq A \colon C \in F\}$ on A we have the natural homeomorphisms:

$$C_F \approx \mathbb{R}^B \times C_G \approx \mathbb{R} \times \mathbb{R}^B \times C_G \approx \mathbb{R} \times C_F. \qquad \square$$

For our purposes we will use the special filters described in [5]. Their construction is as follows. Let 2^n be the set of all functions from $\{0, 1, \ldots, n-1\}$ into $\{0, 1\}$ for $n = 1, 2, \ldots$. Let us put $T = \bigcup\{2^n \colon n = 1, 2, \ldots\}$. For each function $x \colon \omega \to \{0, 1\}$ we define the branch B_x in T as $\{x|n \in 2^n \colon n = 1, 2, \ldots\}$, where $x|n$ denotes the restriction of the function x to the set $\{0, 1, \ldots, n-1\}$. Let A be a subset of the Cantor set 2^ω. The filter F_A on the countable set T is generated by the family $\{T\setminus(B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_n} \cup S) \colon n \geq 1,\ x_i \in A$ and S is a finite subset of $T\}$.

The spaces X_i and Y_i from Theorem 1.1 are of the form N_{F_A} for some suitable analytic or coanalytic subset A of 2^ω. We will use the following two properties of the spaces $C_p(N_{F_A})$ proved in [5]:

Lemma 2.2 [5, Theorem 2.1]. *For every subset A of 2^ω the space $C_p(N_{F_A})$ contains a closed copy of A.*

Lemma 2.3 [5, Theorem 4.1]. *For every analytic (coanalytic) set $A \subseteq 2^\omega$ the filter F_A and the space $C_p(N_{F_A})$ are analytic (coanalytic).*

3. Property (∗)

We say that the space X has the property (∗) if every analytic subset of X is contained in some σ-compact subset of X.

Lemma 3.1. *If the spaces X_i have the property* (∗), *for $i = 1, \ldots, n$, then the product $X_1 \times \cdots \times X_n$ also has* (∗).

Proof. Follows easily from the fact that the projection of an analytic set is analytic. □

Lemma 3.2. *Let the spaces X_i be compact metrizable and the spaces $Y_i \subseteq X_i$ be coanalytic, for $i = 1, 2, \ldots, n$. If all spaces Y_i have the property* (∗) *then the space*

$$A = \left\{(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i \colon (\exists i \leq n)[x_i \in Y_i]\right\}$$

also has (∗).

Proof. An induction on n reduces our proof to the case $n = 2$. Let Z be an analytic subset of $A = Y_1 \times X_2 \cup X_1 \times Y_2$. By our assumptions on X_i and Y_i the sets

$$Z_1 = Z \cap (X_1 \setminus Y_1) \times X_2 \subseteq X_1 \times Y_2$$

and

$$Z_2 = Z \cap X_1 \times (X_2 \setminus Y_2) \subseteq Y_1 \times X_2$$

are also analytic. Applying Lemma 3.1 for $X_1 \times Y_2$ and $Y_1 \times X_2$ we obtain σ-compact sets B_1 and B_2 such that $Z_1 \subseteq B_1 \subseteq X_1 \times Y_2$ and $Z_2 \subseteq B_2 \subseteq Y_1 \times X_2$. The set $Z_3 = Z \setminus (B_1 \cup B_2) \subseteq Y_1 \times Y_2$ is analytic and again by Lemma 3.1 we can find a σ-compact set B_3 such that $Z_3 \subseteq B_3 \subseteq Y_1 \times Y_2$. Finally, we have $Z \subseteq B_1 \cup B_2 \cup B_3 \subseteq A$. $\square$

We shall prove the following facts about the filters F_A described in Section 2:

Lemma 3.3. *Let A be a subset of 2^ω satisfying $(*)$. Then the filter F_A has the property $(*)$.*

Proof. Let $J_A = \{T \setminus X : X \in F_A\}$ be the dual ideal. Since J_A and F_A are homeomorphic it is enough to show $(*)$ for J_A.

Let $f: 2^\omega \times 2^\omega \to 2^T$ be the map defined by

$$f(x, y) = \{x|n \in T : y(n) = 1, n \in \omega\} \quad \text{for } x, y \in 2^\omega.$$

The map f is continuous. Let $E = \{y \in 2^\omega : y^{-1}(1) \text{ is finite}\}$. We denote $f(2^\omega \times 2^\omega)$ by S (S is the family of subsets of branches B_x, $x \in 2^\omega$) and $S \cap J_A$ by S_A. We shall verify the property $(*)$ for S_A.

Let $Z \subseteq S_A$ be analytic. Observe that $f^{-1}(Z)$ is an analytic subset of $2^\omega \times 2^\omega$ which is contained in $A \times 2^\omega \cup 2^\omega \times E$. Since the set $f^{-1}(Z) \setminus 2^\omega \times E$ is also analytic, from the property $(*)$ for $A \times 2^\omega$ (see Lemma 3.1) it follows that $(f^{-1}(Z) \setminus 2^\omega \times E) \subseteq B$ for some σ-compact subset B of $A \times 2^\omega$. Now, $f(B \cup 2^\omega \times E)$ is a σ-compact subset of S_A containing Z.

For every $n \geqslant 1$ consider the continous map $u_n : S^n \to 2^T$ defined by

$$u_n(X_1, \ldots, X_n) = X_1 \cup \cdots \cup X_n \quad \text{for } X_i \in S.$$

One can easily verify that $J_A \subseteq \bigcup_{n \geqslant 1} u_n(S^n)$ and $u_n^{-1}(J_A) = (S_A)^n$.

Let Z be an analytic subset of J_A. Since $u_n(S^n)$ is compact $Z \cap u_n(S^n)$ is also analytic, so $u_n^{-1}(Z)$ is analytic. We have $u_n^{-1}(Z) \subseteq (S_A)^n$ and using $(*)$ for $(S_A)^n$ we can find a σ-compact subset B_n of $(S_A)^n$ containing $u_n^{-1}(Z)$. The set $B = \bigcup_{n \geqslant 1} u_n(B_n)$ is σ-compact and contains $\bigcup_{n \geqslant 1} (Z \cap u_n(S^n)) = Z$. $\square$

Lemma 3.4. *Let A be a coanalytic subset of 2^ω with property $(*)$ and let $B = 2^\omega \setminus A$. Then every analytic subset of $2^T \setminus F_B$ is contained in some absolute $G_{\delta\sigma}$ subset of $2^T \setminus F_B$.*

Proof. Similarly as in Lemma 3.3 we will consider the dual ideals J_A and J_B instead of the filters F_A and F_B. We denote J_{2^ω} by J. Let S and S_A be as in the proof of Lemma 3.3 and let $H = \{X \subseteq T : X \text{ is finite}\}$.

For every $X \in S \setminus H$ let $p(X)$ be the unique $x \in 2^\omega$ such that $X \subseteq B_x$. The map $p: S\setminus H \to 2^\omega$ is continuous. By $<$ we denote the order on 2^ω induced by the standard embedding of 2^ω into $\mathbb{R}$. For every $n \geq 1$ and $E \in H$ we put

$$R_n^E = \{(X_1, X_2, \dots, X_n) \in (S\setminus H)^n : p(X_1) < p(X_2) < \cdots < p(X_n),$$
$$X_j \cap B_{p(X_i)} = \emptyset \text{ for } i < j \leq n$$
$$\text{and } X_i \cap E = \emptyset \text{ for } i \leq n\}.$$

The set R_n^E is open in $(S\setminus H)^n$ and therefore an absolute G_δ-set. We define the map $u_n^E : R_n^E \to 2^T$ by the formula:

$$u_n^E(X_1, \dots, X_n) = X_1 \cup \cdots \cup X_n \cup E.$$

One can easily verify that u_n^E is a homeomorphism of R_n^E onto $u_n^E(R_n^E)$ and $J = H \cup \bigcup \{u_n^E(R_n^E): n \geq 1, E \in H\}$.

Let Z be an analytic subset of $2^T \setminus J_B$. We put $Z' = Z \cap J$. Obviously $Z' \subseteq J\setminus H$.

Fix $n \geq 1$ and $E \in H$. The set $Z' \cap u_n^E(R_n^E)$ is analytic, therefore $(u_n^E)^{-1}(Z')$ is also analytic.

Let $M_n = \{(X_1, \dots, X_n) \in S^n : (\exists i)[X_i \in S_A]\}$, then by Lemmas 2.3, 3.2 and the property $(*)$ for S_A (see the proof of Lemma 3.3) M_n has $(*)$. Since $(u_n^E)^{-1}(Z')$ is contained in M_n, there exists a σ-compact set D_n^E such that $(u_n^E)^{-1}(Z') \subseteq D_n^E \subseteq M_n$. R_n^E being an absolute G_δ-set, the set $R_n^E \cap D_n^E$ is an absolute $G_{\delta\sigma}$-set. Observe that $u_n^E(R_n^E \cap M_n) \subseteq J\setminus J_B$, therefore the set

$$D = (2^T \setminus J) \cup \bigcup \{u_n^E(D_n^E \cap R_n^E): n \geq 1 \text{ and } E \in H\}$$

is an absolute $G_{\delta\sigma}$-set and $Z \subseteq D \subseteq 2^T \setminus J_B$. $\square$

4. Zero-dimensional subsets of the spaces C_F

In this section we prove the following result about zero-dimensional subsets of the spaces C_F:

Lemma 4.1. *Let F be a filter on the countable set T. Every closed zero-dimensional subset of the space C_F is homeomorphic to a closed subset of the product F^ω.*

Proof. We will identify the set T with the set of positive integers $\mathbb{N}$. Let X be a closed zero-dimensional subset of C_F and d be any totally bounded metric on X. By $\mathbb{Q}$ we denote the space of rationals.

First, we shall find sets $A_n \subseteq \mathbb{Q}$, closed and discrete in $\mathbb{R}$, and construct a closed embedding of X into $C_F \cap \prod_{n\in\mathbb{N}} A_n$.

For every $n \geq 1$ we construct a continuous function $g_n : X \to \mathbb{Q}$ with the following properties:

(1) $(\forall x \in X\ (x = (x_i)_{i \in \mathbb{N}}))[|g_n(x) - x_n| \leq 1/n]$,

(2) $(\forall x, y \in X)[g_n(x) = g_n(y) \Rightarrow d(x, y) \leq 1/n]$,

(3) $A_n = g_n(X)$ is closed and discrete in $\mathbb{R}$.

To obtain g_n consider the cover $\{V_k : k \text{ integer}\}$ of X, where $V_k = \{(x_i)_{i \in \mathbb{N}} \in X : x_n \in (k/(2n), (k+2)/(2n))\}$ and take a refinement $\{U_i : i \in \omega\}$ consisting of disjoint clopen sets of diameter less than $1/n$. Since (X, d) is totally bounded we can assume that for every k the set $\{i : U_i \subseteq V_k\}$ is finite. For every $i \in \omega$ we take k such that $U_i \subseteq V_k$ and choose $q_i \in \mathbb{Q} \cap (k/(2n), (k+2)/(2n))$ in such a way that $q_i \neq q_j$ for $i \neq j$. Finally, q_i is the value of g_n on U_i.

Let $g = \Delta_{n \in \mathbb{N}} g_n : X \to \mathbb{Q}^{\mathbb{N}}$ be a diagonal map. By (1) we have $\lim_n (g_n(x) - x_n) = 0$ for every $x = (x_n)_{n \in \mathbb{N}} \in X$. This implies that $g(x) \in C_F$. From (2) it follows that g is one-to-one. To complete the proof of that g is a closed embedding it suffices to show that for every closed subset A of X its image $g(A)$ is closed in C_F.

Let $x^k = (x_n^k)_{n \in \mathbb{N}} \in A$ and $z = (z_n)_{n \in \mathbb{N}} \in C_F$ be such that $g(x^k) \to z$, i.e.,

(4) $(\forall n)[\lim_k g_n(x^k) = z_n]$.

From (1) it follows that for every $n \geq 1$ the sequence $(x_n^k)_{k \in \omega}$ is bounded and we can assume that $\lim_k x_n^k = x_n$ for some $x_n \in \mathbb{R}$. By (1) and (4), $|z_n - x_n| \leq 1/n$, therefore the sequence $(x_n)_{n \in \mathbb{N}}$ belongs to C_F and consequently $(x_n)_{n \in \mathbb{N}} \in A$. Obviously $g_n(x_n) = z_n$, i.e., $g(A)$ is closed in C_F.

To finish the proof we will show that there exists a closed embedding $h : C_F \cap \prod_{n \in \mathbb{N}} A_n \to F^\omega$.

Since F is noncompact we can consider each A_n as a closed subset of F. Also, we can identify each A_n with the subset of 2^ω by identifying $\mathbb{Q}$ with a subset of 2^ω, where 0 corresponds to $O = (0, 0, \dots)$. These identifications yield embeddings i and j of $\prod_{n \in \mathbb{N}} A_n$ into $F^{\mathbb{N}}$ and $(2^\omega)^{\mathbb{N}}$ such that the diagonal map $h' = i \Delta j$ is a closed embedding of $\prod_{n \in \mathbb{N}} A_n$ into $F^{\mathbb{N}} \times (2^\omega)^{\mathbb{N}}$. One can easily check that h' maps $\prod_{n \in \mathbb{N}} A_n \cap C_F$ onto $h'(\prod_{n \in \mathbb{N}} A_n) \cap (F^{\mathbb{N}} \times Z_F)$, where $Z_F = \{f : \omega \to 2^\omega : (\forall \varepsilon > 0)(\exists A \in F)(\forall n \in A)[\rho(f(n), O) \leq \varepsilon]\}$ and ρ is a fixed metric on 2^ω. To complete the proof let us observe that Z_F is homeomorphic to F^ω. Indeed: for $f \in Z_F$ let $e(f) = (p_n \circ f)_{n \in \omega}$, where $p_n : 2^\omega \to 2$ is the projection onto the nth coordinate. It is routine to verify that the map $f \mapsto e(f)$ is a homeomorphism of Z_F onto $(C_F \cap 2^{\mathbb{N}})^\omega$. Evidently $C_F \cap 2^{\mathbb{N}}$ is homeomorphic to F. $\square$

5. Proof of Theorem 1.1

First we describe the spaces X_i. Let A_1 (A_2) be an analytic (a coanalytic) subset of 2^ω universal for the analytic (coanalytic) subsets of 2^ω, i.e., A_1 (A_2) contains closed copies of all zero-dimensional analytic (coanalytic) spaces, see [4, 38.V]. Let $F_i = F_{A_i}$ be the filter described in Section 2 and let $X_i = N_{F_i}$, for $i = 1, 2$. By Lemmas 2.2 and 2.3 the space $C_p(X_1)$ is analytic noncoanalytic ($C_p(X_2)$ is coanalytic nonanalytic) and contains closed copies of all zero-dimensional analytic (coanalytic) spaces. In particular every zero-dimensional absolute Borel set is homeomorphic to a closed subset of $C_p(X_i)$, for $i = 1, 2$.

Remark 5.1. Let us notice that in [1] examples were constructed of filters F_n for which the space $C_p(N_{F_n})$ is an absorbing set for the class of projective sets of class n, $n \geqslant 1$, in particular $C_p(N_{F_n})$ contains closed copies of all projective spaces of the class n.

Let B be an uncountable coanalytic subset of 2^ω which does not contain any copy of 2^ω. Since every analytic subset of B is countable, B has the property $(*)$, see Section 3. We put $C = 2^\omega \setminus B$, $Y_1 = N_{F_C}$ and $Y_2 = N_{F_B}$. The following lemmas show that the spaces $C_p(X_i)$ and $C_p(Y_i)$ are topologically different, for $i = 1, 2$:

Lemma 5.2. *Every closed zero-dimensional coanalytic subset of $C_p(Y_1)$ is an absolute $F_{\sigma\delta}$-set.*

Proof. Let $Z \subseteq C_p(Y_1)$ be a closed zero-dimensional coanalytic space. Using Lemmas 2.1 and 4.1 we can regard Z as a closed subset of $(F_C)^\omega$. Let Z' be a closed subset of $(2^T)^\omega$ such that $Z' \cap (F_C)^\omega = Z$. By p_n we denote the projection of $(2^T)^\omega$ onto the nth coordinate and we put $Z_n = p_n^{-1}(2^T \setminus F_C) \cap Z'$. For every $n \in \omega$, Z_n is a coanalytic set and $Z' = Z \cup Z_0 \cup Z_1 \cup \cdots$. By the reduction theorem, see [4, Theorem 1, 39.X] there exist absolute Borel subsets $D_n \subseteq Z_n$, for $n \in \omega$, such that $Z' = Z \cup D_0 \cup D_1 \cup \cdots$. Now, the set $p_n(D_n)$ is an analytic subset of $2^T \setminus F_C$ and by Lemma 3.4 is contained in some absolute $G_{\delta\sigma}$-subset E_n of $2^T \setminus F_C$. Since

$$Z' \setminus Z = Z' \cap \bigcup \{p_n^{-1}(E_n) \colon n \in \omega\},$$

$Z' \setminus Z$ is an absolute $G_{\delta\sigma}$-set. Therefore Z is an absolute $F_{\sigma\delta}$-set. $\square$

Lemma 5.3. *Every closed zero-dimensional analytic subset of $C_p(Y_2)$ is an absolute $F_{\sigma\delta}$-set.*

Proof. Similarly as in the proof of Lemma 5.2 we need to prove that every closed analytic subset $Z \subseteq (F_B)^\omega$ is an absolute $F_{\sigma\delta}$-set. Let Z_n be the projection of Z onto the nth coordinate, $n \in \omega$. Applying Lemma 3.2, for every $n \in \omega$, we obtain a σ-compact set $B_n \subseteq F_B$ containing Z_n. Z is a closed subset of $\prod_{n \in \omega} B_n$ and therefore an absolute $F_{\sigma\delta}$-set. $\square$

Remark 5.4. If we assume the determinacy of analytic sets (see [3, Chapter 7.43] or [6, Chapter 6.G]) then every analytic noncoanalytic (coanalytic nonanalytic) space $C_p(X)$ will contain closed copies of all zero-dimensional analytic (coanalytic) spaces. By the results of [1], in order to prove that under this assumption all analytic noncoanalytic (coanalytic nonanalytic) spaces $C_p(X)$ are homeomorphic it would be sufficient to show that $C_p(X)$ contains closed copies of all analytic (coanalytic) spaces.

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